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Nonlinear Ergodic Theorems for Asymptotically Nonexpansive Mappings in Banach Spaces Satisfying Opial's Condition

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The following theorem is proven: If E is a uniformly convex Banach space satisfying Opial's condition, C is a nonempty bounded closed convex subset of E, and $T: C \to C$ is an asymptotically nonexpansive mapping then the iterates $\{T^n x\}$ are weakly almost-convergent to a fixed point of T. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let C be a nonempty subset of a Banach space E and T be a mapping of C into itself. T is said to be asymptotically nonexpansive [6] if for $x, y \in C$

$$\|T^i x - T^i y\| \leqslant k_i \cdot \|x - y\| \qquad \text{for} \quad i = 1, 2, ...,$$

where

$$|||T^i||| = k_i = \sup\left(\frac{\|T^i x - T^i y\|}{\|x - y\|} : x \neq y,\ x, y \in C\right)$$

is the Lipschitz constant of T^i and $k_i \downarrow 1$. In particular if $k_i = 1$, $i = 1, 2, ...$, then T is said to be nonexpansive. The class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings [6].

The following is known as the Mean Ergodic Theorem for nonexpansive mappings. It is in essence due to J. B. Baillon.

THEOREM (MET). *Let C be a bounded closed convex subset of a uniformly convex Banach space with Fréchet differentiable norm, and let $T: C \to C$ be nonexpansive. For $x \in C$ let*

$$S_n x = \frac{1}{n} \sum_{i=0}^{n-1} T^i x.$$

Then $\{S_n x\}$ converges weakly to a fixed point of T.

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The above theorem was proved first for a Hilbert space by J. B. Baillon [1]. H. Brezis and F. E. Browder [3] generalized Baillon's results to more general summation methods

$$A_n x = \sum_{k=0}^{\infty} a_{nk} T^k x, \qquad n = 1, 2, ...,$$

where $A = [a_{nk}]_{n,k \geqslant 0}$ is an arbitrary strongly ergodic matrix:

(1°) $\bigwedge_{n,k} a_{nk} \geqslant 0$,

(2°) $\bigwedge_k \lim_{n \to \infty} a_{nk} = 0$,

(3°) $\bigwedge_n \sum_{k=0}^{\infty} a_{nk} = 1$,

(4°) $\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{n,k+1} - a_{nk}| = 0$.

The concept of almost-convergence is due to G. G. Lorentz (1948): a sequence $\{x_n\} \subset E$ is said to be (weak-) almost-convergent to a point $x_0 \in E$ iff

$$\text{(weak-)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} x_{k+i} = x_0,$$

uniformly in $i = 0, 1, 2, ...$.

In a Banach space it is equivalent, that for each strongly ergodic matrix $A = [a_{nk}]_{n,k \geqslant 0}$ we have

$$\text{(weak-)} \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} \cdot x_k = x_0.$$

S. Reich [13] extended Baillon's results to uniformly convex Banach spaces with Fréchet differentiable norms. R. E. Bruck [4] simplified the original argument of Reich.

N. Hirano and W. Takahashi [9], at first, extend Baillon's theorem for asymptotically nonexpansive mappings in real Hilbert spaces. This result has been generalized by B. E. Rhoades [14]. Recently, M. Krüppel [10] generalized Hirano–Takahashi's result to uniformly convex Banach spaces with Fréchet differentiable norms.

Ergodic theorems for nonexpansive mappings in uniformly convex Banach spaces which satisfy Opial's condition have been studied in the papers [2, 8, 12, 15].

In this note we give an ergodic theorem for asymptotically nonexpansive mappings in B-convex Banach spaces satisfying Opial's condition.

E is a B-convex Banach space iff l_1 is not finitely reprezentable in E. For example, l_∞, c_0 are not B-convex. Not all B-convex spaces are reflexive, but every uniformly convex Banach space is B-convex [16].

Notation. Weak convergence of a sequence will be denoted by $x_n \rightharpoonup x$, strong convergence by $x_n \to x$. The set of fixed points of a mapping T will be denoted by $F(T)$.

2. Behavior of Asymptotically Nonexpansive Mappings

Denote by Γ the set of all strictly increasing convex functions $\gamma\colon [0, +\infty) \to [0, +\infty)$ with $\gamma(0)=0$. Let $C \subset E$. A mapping $T\colon C \to C$ is said to be of type (γ) iff

$$\bigvee_{\gamma\in\Gamma} \bigwedge_{x,\, y\in C} \bigwedge_{0\leqslant c\leqslant 1} \gamma(\|cTx+(1-c)\,Ty - T(cx+(1-c)\,y)\|) \leqslant \|x-y\| - \|Tx-Ty\|.$$

If E is a uniformly convex Banach space then every nonexpansive mapping $T\colon C\to C$ is of type (γ); moreover, γ can be chosen to depend only on diam C and not on T [4].

Without loss of generality we assume in this paper that $0 \in C$ and diam $C \leqslant 1$.

Let $C\subset E$ and $T\colon C\to C$ be a mapping. We shall call a sequence $\{x_n\}$ in C a $(*)$-property of T if

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0.$$

Lemma 2.1. *If $T\colon C\to C$ is an asymptotically nonexpansive mapping and $\{x_n\}$, $\{y_n\}$ are sequences with $(*)$-property of T then $\{\|x_n - y_n\|\}$ converges.*

Proof. Put

$$a_m = \limsup_{p\to\infty} \|x_p - T^m x_p\|,\ b_m = \limsup_{p\to\infty} \|y_p - T^m y_p\|,$$

and k_m be the Lipschitz constant of T^m, $m=1, 2, \ldots$. Then $a_m\to 0$, $b_m \to 0$, $k_m \downarrow 1$ as $m\to +\infty$ and for arbitrary m, n holds

$$\begin{aligned}\|x_{n+m} - y_{n+m}\| &\leqslant \|x_{n+m} - T^m x_{n+m}\| + \|T^m x_{n+m} - T^m y_{n+m}\| \\ &\quad + \|T^m y_{n+m} - y_{n+m}\| \\ &\leqslant a_m + b_m + k_m \cdot \|x_{n+m} - y_{n+m}\|.\end{aligned}$$

For every m, let $\{n_i^{(m)}\}_i$ be such a subsequence that

$$\|x_{n_i^{(m)}+m} - y_{n_i^{(m)}+m}\| \to \liminf_{p\to\infty} \|x_p - y_p\| \qquad \text{as} \quad i\to +\infty$$

and

$$\|x_{n_m^{(m)}+m} - y_{n_m^{(m)}+m}\| \to \limsup_{p \to \infty} \|x_p - y_p\| \qquad \text{as} \quad m \to +\infty.$$

Then

$$\|x_{n_i^{(m)}+m} - y_{n_i^{(m)}+m}\| \leqslant a_m + b_m + k_m \cdot \limsup_{p \to \infty} \|x_p - y_p\|$$

for $i > N$, and

$$\|x_{n_m^{(m)}+m} - y_{n_m^{(m)}+m}\| \leqslant a_m + b_m + k_m \cdot \liminf_{p \to \infty} \|x_p - y_p\|$$

for $m > N$. Taking the lim sup as $m \to +\infty$ on both sides, we get

$$\limsup_{p \to \infty} \|x_p - y_p\| \leqslant \liminf_{p \to \infty} \|x_p - y_p\|.$$

M. Krüppel [10] proved the following

THEOREM 2.2. *Let E be a B-convex Banach space, $C \subset E$ be a bounded closed convex and weakly compact, $T\colon C \to C$ is an asymptotically nonexpansive mapping such that $|||T^n|||^{-1} \cdot T^n$ is of type (γ) for $n = 1, 2, \ldots$, and let*

$$y_n = A_n x = \sum_{k=0}^{\infty} a_{nk} \cdot T^k x, \qquad x \in C,\ n = 1, 2, \ldots,$$

where $A = [a_{nk}]_{n,k \geqslant 0}$ is a strongly ergodic matrix. Then

$$\lim_{m \to \infty} \limsup_{n \to \infty} \|y_n - T^m y_n\| = 0.$$

3. Banach Spaces Satisfying Opial's Condition

We say that a Banach space E satisfies the Opial's condition [12] if for each sequence $\{x_n\} \subset E$ weakly convergent to a point x, and for all $y \neq x$

$$\liminf_{n \to \infty} \|x_n - y\| > \liminf_{n \to \infty} \|x_n - x\|. \tag{1}$$

It is known that (1) is equivalent to the analogous condition obtained by replacing lim inf by lim sup.

The examples of a Banach space which satisfies the Opial's condition are Hilbert spaces and all spaces l^p $(1 \leqslant p < +\infty)$. On the other hand $L^p[0, 2\pi]$ with $1 < p \neq 2$ fails to satisfy Opial's condition [11, 12].

LEMMA 3.1 [10]. *Let C be a bounded closed convex subset of a B-convex Banach space E. Assume that $S: C \to C$ is a nonexpansive mapping of type (γ). If $y_n \rightharpoonup y$ ($y_n, y \in C$, $n = 1, 2, \ldots$) then there exists a function $g \in \Gamma$ such that*

$$g(\|y - Sy\|) \leqslant \liminf_{n \to \infty} \|y_n - Sy_n\|.$$

THEOREM 3.2. *Let E be a B-convex Banach space satisfying Opial's condition and $C \subset E$ be a bounded closed convex and weakly compact. Let $T: C \to C$ be an asymptotically nonexpansive mapping such that $|||T^n|||^{-1} \cdot T^n$ is of type (γ) for $n = 1, 2, \ldots$. If $A = [a_{nk}]_{n,k} \geqslant 0$ is an arbitrary strongly ergodic matrix, than for each x in C the sequence*

$$y_n = A_n x = \sum_{k=0}^{\infty} a_{nk} \cdot T^k x, \qquad n = 1, 2, \ldots,$$

converges weakly to a fixed point of T.

Proof. First we show that any weak subsequential limit y of $\{y_n\}$ is a fixed point of T. By the Theorem 2.2,

$$u_m := \limsup_{n \to \infty} \|y_n - T^m y_n\| \to 0 \qquad \text{as} \quad m \to +\infty.$$

Because $|||T^m||| \geqslant 1$ for every $m = 1, 2, \ldots$, the mapping

$$S_m = |||T^m|||^{-1} \cdot T^m : C \to C$$

is nonexpansive. Let $y_{n_i} \rightharpoonup y$ as $i \to +\infty$. By Lemma 3.1, there exists a function $g \in \Gamma$ such that

$$\begin{aligned} g(\|y - S_m y\|) &\leqslant \liminf_{i \to \infty} \|y_{n_i} - |||T^m|||^{-1} \cdot T^m y_{n_i}\| \\ &\leqslant \limsup_{i \to \infty} [|||T^m|||^{-1} \cdot \|y_{n_i} - T^m y_{n_i}\| \\ &\qquad + (1 - |||T^m|||^{-1}) \cdot \|y_{n_i}\|] \\ &\leqslant |||T^m|||^{-1} \cdot u_m + (1 - |||T^m|||^{-1}). \end{aligned}$$

Obviously,

$$\begin{aligned} \|y - T^m y\| &\leqslant |||T^m||| \cdot [\|y - |||T^m|||^{-1} \cdot T^m y\| + (1 - |||T^m|||^{-1}) \cdot \|y\|] \\ &= (|||T^m||| - 1) \cdot \|y\| + |||T^m||| \cdot \|y - |||T^m|||^{-1} \cdot T^m y\| \\ &\leqslant (|||T^m||| - 1) + |||T^m||| \cdot g^{-1}(|||T^m|||^{-1} \cdot u_m + (1 - |||T^m|||^{-1})). \end{aligned}$$

By $u_m \to 0$ and $|||T^m||| \downarrow 1$ as $m \to +\infty$ it follows

$$\lim_{m \to \infty} \|y - T^m y\| = 0,$$

i.e., $T^m y \to y$ as $m \to +\infty$. But T is continuous, so we have

$$Ty = T(\lim_{m \to \infty} T^m y) = \lim_{m \to \infty} T^{m+1} y = y.$$

Thus $F(T) \neq \varnothing$ and all weak subsequential limits of $\{y_n\}$ are in $F(T)$. By Lemma 2.1 and Lemma 2.2 for any fixed point $p \in F(T)$, the sequence $\{\|y_n - p\|\}$ is convergent. Suppose that $y_{n_i} \rightharpoonup y$ as $i \to +\infty$ and $y_{m_i} \rightharpoonup y'$ as $i \to +\infty$ and $y \neq y'$. Since E satisfies Opial's condition we have

$$\liminf_{i \to \infty} \|y_{n_i} - y\| < \liminf_{i \to \infty} \|y_{n_i} - y'\|,$$

i.e.,

$$\lim_{n \to \infty} \|y_n - y\| < \lim_{n \to \infty} \|y_n - y'\|.$$

Similarly, replacing the role of $\{y_{n_i}\}$ by $\{y_{m_i}\}$ we derive

$$\lim_{n \to \infty} \|y_n - y'\| < \lim_{n \to \infty} \|y_n - y\|,$$

which is a contradiction. Hence $y = y'$. Thus $\{y_n\}$ possesses only one weak limiting point, i.e., $\{y_n\}$ converges weakly to a fixed point of T.

COROLLARY 3.3. *If E is a uniformly convex Banach space satisfying Opial's condition and C a nonempty bounded closed convex subset of E, then for each asymptotically nonexpansive mapping $T: C \to C$ the sequence of iterates $\{T^n x\}$ weakly almost-convergent to a fixed point of T.*

4. REMARK

Let E be a Banach space, E^* its dual space, and $\langle f, x \rangle$ the value of the linear functional $f \in E^*$ at the element $x \in E$. Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, strictly increasing, and $\phi(0) = 0$. A mapping $J_\phi: E \to E^*$ is called a duality mapping of E into E^* with the gauge function ϕ if $\langle J_\phi(x), x \rangle = \|x\| \cdot \|J_\phi(x)\|$ and $\|J_\phi(x)\| = \phi(\|x\|)$ for all $x \in E$.

J. P. Gossez and E. Lami Dozo [7] have shown that for any normed linear space E, the existence of a weakly sequentially continuous duality map implies that E satisfies the Opial's condition. No converse implication holds even when E and E^* are supposed to be uniformly convex.

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